

Statistical Effects and the Black Hole/D-brane Correspondence

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(February 1, 2008)

The horizon area and curvature of three-charge BPS black strings are studied in the D-brane ensemble for the stationary black string. The charge distributions along the string are used to translate the classical expressions for the horizon area and curvature of BPS black strings with waves into operators on the D-brane Hilbert space. Despite the fact that any ‘wavy’ black string has smaller horizon area and divergent curvature, the typical values of the horizon area and effects of the horizon curvature in the D-brane ensemble deviate negligibly from those of the original stationary black string in the limit of large integer charges. Whether this holds in general will depend on certain properties of the quantum bound states.

I. INTRODUCTION

The correspondence between various classical black hole solutions and ensembles of D-brane bound states is by now quite well established [1–10]. What is not so clear is the exact meaning of this correspondence. For example, controversy continues about the implications for information loss. In addition, Myers [11] has recently raised the issue of whether the ensemble nature of the quantum state is essential; that is, whether it might be impossible to establish a similar correspondence between a classical black hole and a pure quantum state. We address this issue below and, while we will not settle it conclusively, a framework is provided for its further study.

Let us recall the typical form of results on the black hole/D-brane correspondence. For definiteness, we consider the ‘black holes’ of [4,12,13] with a 4+1 asymptotically flat space and an internal T^5 ; such objects correspond to bound states of D-onebranes and D-fivebranes with internal momentum. In particular, we will be interested in the case where one internal S^1 is large and the solution may be thought of as a black *string*, with the momentum flowing around this S^1 . One first considers the classical black string, which is defined by a set of ‘macroscopic parameters’ $\{Q_i\}$ – charges [1,2], or perhaps long wavelength charge distributions [14,15]. Three important such parameters are the total one- and five-brane charges (Q_1, Q_5) and the momentum quantum number (N). The black string solution is chosen to have no structure on smaller scales; that is, any remaining parameters $\{q_a\}$ which describe the black string are set to zero. Examples of such $\{q_a\}$ are the short wavelength components of the charge distributions. One then uses the macroscopic parameters to define an ensemble of D-brane states. The macroscopic parameters of such states are required to agree with the classical black hole, but their ‘microscopic structure’ is not constrained. One computes some property of the ensemble such as the entropy [1–6] or the scattering cross section [7–10], and compares it with the classical black hole results, finding agreement in the limit of large charges.

The approach below is somewhat different. Our goal is to identify a quantum operator A whose classical limit yields the horizon area of the black hole and a family of operators $R_{\alpha\beta\gamma\delta}(x)$ whose classical limits give the curvature tensor in some coordinate system. These will involve both the macroscopic $\{Q_i\}$ and the microscopic $\{q_a\}$. We then use these operators to study statistical mechanical effects on the horizon area and curvatures. A quantum treatment is necessary since the bound states form a field theory with the corresponding divergences in classical statistical mechanics.

We begin by taking seriously the conjecture that one can establish a correspondence between (suitable limits of) individual D-brane states and classical black hole solutions. This is consistent with the usual picture of a classical state corresponding to a limit of quantum states, as $\hbar \rightarrow 0$. In this case, any function on the space of classical solutions should arise as the classical limit of an operator on the quantum state space. We will use the classical expressions for A and $R_{\alpha\beta\gamma\delta}(x)$ in terms of the charge distributions to motivate a choice of quantum operator. The expected values of A and R will then be computed in the D-brane ensemble and compared with the stationary classical black string. Note that we do not require the weakly coupled D-branes to have a horizon of area A or curvature $R_{\alpha\beta\gamma\delta}(x)$ in any physical sense; the point is simply that, if the D-brane ensemble successfully characterizes the charge distribution fluctuations of the black string, such operators will provide the effects of these fluctuations on the area and curvatures. We consider the case where the ‘macroscopic parameters’ $\{Q_i\}$ are just the integer charges while the microscopic parameters $\{q_a\}$ include all information about the distribution of this charge.

One typically expects that any unconstrained quantity (such as q_a) will average to zero and that fluctuations are small in the thermodynamic limit. As a result, the expected values of A and R should not differ significantly from their values for the original black string. However, the area operator has the property that it is equally sensitive to fluctuations on all distance scales. In fact, the expected value of A will receive a correction from each q_a that is nonzero and these corrections are additive. The curvature depends even more dramatically on many of the microscopic parameters q_a . As shown in [16], any inhomogeneities in the distribution of longitudinal momentum along the string results in a singular curvature at the horizon. Thus, this curvature in general diverges when $q_a \neq 0$. However, the singularity is weak and its physical effects are finite. The point of our study is to verify that the fluctuations are small enough to have negligible effects even on these sensitive characteristics. We will find that the resulting deviations are suppressed by powers of Q_1 and Q_5 , but not by powers of N . As a result, the horizon area and curvatures of a stationary black string correspond well to $\langle A \rangle$ and $\langle R \rangle$ not just for the D-brane ensemble as a whole, but also for a typical state.

In the course of our argument, a number of subtleties will arise in relating the classical black strings to quantum states. They are connected to the question of how to choose the parameters q_a and the function $A(Q_i, q_a)$. A working hypothesis is stated, and arguments are given in its favor. A complete understanding of this issue is likely to elucidate many features of the black hole/D-brane correspondence.

The structure of this paper is as follows. Section II reviews certain properties of the classical supergravity solutions corresponding to six dimensional BPS black strings, providing a slight generalization of previous work. These results will be used to define the D-brane ‘area operator’ and ‘curvature operator’ as it is in terms of such asymptotic properties that the black hole/D-brane correspondence seems most direct. Section III studies the area operator while section IV studies the curvatures. We close with a discussion of results and certain subtleties in section V.

II. THE SPACE OF CLASSICAL SOLUTIONS

Recall that our goal is to use the classical expression for the horizon area and curvature in terms of the charge densities to define quantum operators on the D-brane Hilbert space. As a result, we should consider a space of classical solutions and a set of D-brane states such that each classical solution is associated with a suitable limit of our D-brane states (or ensembles of such states) and such that each D-brane state takes part in this limit.

We focus on the class of solutions known as six-dimensional BPS black strings with traveling waves [12,13,17]. Such solutions generalize the more standard stationary and translationally invariant solution [18,19] and have been shown to correspond to ensembles of D-brane states [14,15]. In particular, the associated D-brane ensembles are *subsets* of the ensemble usually used to describe the stationary black string [4]. As a result, despite the singularities found in [16], such solutions must be included in our discussion.

Ideally, we would like to consider the space of *all* BPS solutions. Then, assuming that A) the resulting horizon area and curvatures can be written as a function of the asymptotic charge distributions and B) a correspondence between the asymptotic charge distributions of the classical BPS solutions and the D-brane states can be made, we would then use these charges to construct operators on the D-brane Hilbert space. However, to even state precisely just what is meant by ‘all BPS solutions’ would require a detailed understanding of the classical limit of all BPS string states. As a result, we shall take the much more modest approach of using a set of classical solutions which has already been studied in the literature to refine the area only a single step beyond the most naive $A = (4G_N)2\pi\sqrt{Q_1Q_5N}$, and similarly for the curvatures.

Specifically, we consider the BPS black strings of [14,15], generalized to allow spatially varying moduli and the simultaneous presence of waves carrying angular momentum, those carrying momentum around the T^4 , and the usual ‘longitudinal waves.’ Recall that the low energy action for type IIB string theory (in the Einstein frame) contains the terms

$$S = \frac{1}{16\pi G_{10}} \int dx^{10} \sqrt{-g} \left(R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{12}e^\phi \mathcal{H}^2 \right) \quad (2.1)$$

where ϕ is the dilaton and \mathcal{H} is the Ramond-Ramond three form. We are interested in solutions for which there are 4 + 1 asymptotically flat dimensions (x^a, t) , one ‘large’ internal S^1 (z) of length L (at infinity), and four ‘small’ internal dimensions forming a T^4 (y^i) of volume $V = (2\pi)^4 V$ (at infinity). There is a D-onebrane charge in the large S^1 direction and a D-fivebrane charge around the full internal T^5 . The moduli satisfy $L, V^{1/4} \gg l_{string}$ so that the classical supergravity description is appropriate. We require the horizon to be ‘hole-shaped’ ($S^3 \times \mathbf{R}$) as viewed from the asymptotically flat space, and to have topology $S^3 \times T^5 \times \mathbf{R}$ in the full ten dimensional solution. We consider solutions in which the only complications added to the simplest black string solutions are certain ‘waves’ which run around the large S^1 . In particular, we take our solutions (in the Einstein frame) to be of the form

$$ds^2 = H_1^{1/4} H_5^{3/4} \left[\frac{du}{H_1 H_5} (-dv + K du + 2A_i dy^i + 2A_a dx^a) + \frac{dy^i dy_i}{H_5} + dx^a dx_a \right] \quad (2.2)$$

$$e^{-2\phi} = \frac{H_5}{H_1} \quad (2.3)$$

$$\mathcal{H}_{auv} = H_1^{-2} \partial_a H_1, \quad \mathcal{H}_{aub} = 2\partial_{[a} A_{b]}, \quad \mathcal{H}_{iuj} = 2\partial_{[i} A_{j]}, \quad \mathcal{H}_{abc} = -\epsilon_{abcd} \partial^d H_5 \quad (2.4)$$

where $H_1 = 1 + \frac{r_1^2}{r^2}$, $H_5 = 1 + \frac{r_5^2}{r^2}$, $dx^a dx_a = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\psi^2)$ and K , A_a , and A_i are functions of x, y, u but not v . The indices i, a are raised and lowered with the Euclidean metrics δ_{ij} and δ_{ab} and both run over $\{1, 2, 3, 4\}$. Here the physical values r_1, r_5 of the one- and fivebrane charges as related to the quantized integer charges Q_1 and Q_5 through

$$Q_1 = \frac{V r_1^2}{g}, \quad Q_5 = \frac{r_5^2}{g} \quad (2.5)$$

where g is the string coupling constant. The horizon lies at $r = 0$.

Note that the asymptotic values of K , A_i , and A_a determine the total momenta carried by the black string. Due to the null translational symmetry, it seems natural to interpret K , A_i , and A_a as defining momentum *densities* in the z and x^i directions as functions of u , an angular momentum density, and even higher multipole moments of the momentum and angular momentum densities¹ on the S^3 and T^4 . We take

$$\begin{aligned} K &= \frac{p(u)}{r^2} \\ A_i &= -\frac{p_i(u)}{r^2} \\ A_a dx^a &= \frac{\gamma(u)}{r^2} (\sin^2 \theta d\varphi - \cos^2 \theta d\psi) \end{aligned} \quad (2.6)$$

so that the momentum and angular momentum densities are

$$\frac{dP_z}{du} = \kappa^{-2} p \quad (2.7)$$

$$\frac{dP_i}{du} = \kappa^{-2} p_i \quad (2.8)$$

$$\frac{dJ_\varphi}{du} = -\frac{dJ_\psi}{du} = \kappa^{-2} \gamma(u) \quad (2.9)$$

$$\frac{dE}{du} = \kappa^{-2} (r_1^2 + r_5^2 + p) \quad (2.10)$$

where we have introduced

$$\kappa^2 = \frac{4G_{10}}{\pi V} = \frac{2\pi g^2}{V}. \quad (2.11)$$

This is essentially the same as the case studied in [14,15], though our parametrization of the waves is slightly different. Note that the momentum *densities* are conserved on this class of solutions since $\frac{\partial}{\partial v}$ is a (null) Killing field.

¹See, for example, [20] for a construction of certain multipole moments in stationary spacetimes.

The reader may notice that we have *not* allowed our black string to carry what were called ‘external waves’ in [14,15]. These are the waves which give the string a nonzero density of momentum in the x^a directions and which can be thought of as transverse oscillations of the black string. Such waves have been excluded since they involve oscillations of both the one-brane and five-brane degrees of freedom (the black string oscillates as a whole). Given that a complete description of the five-brane degrees of freedom is beyond the scope of this work, leaving out the external waves is a natural choice.

Since we have allowed only low order multipole terms in (2.6), a slight extension of [14,15] shows that the metric is C^0 at the horizon so that the horizon area is well defined. It takes the form

$$A = 2\pi^2 r_1^2 r_5^2 \mathcal{V} \int du \sigma(u) \quad (2.12)$$

where $\sigma(u)$ is a periodic solution of

$$r_1^{-2} r_5^{-2} [p(u) - p_i(u) p^i(u)/r_1^2 - \gamma^2(u)/r_1^2 r_5^2] = \frac{\partial}{\partial u} \sigma + \sigma^2. \quad (2.13)$$

Thus, for this class of solutions, we have expressed the horizon area in terms of the asymptotic charge distribution. For ‘slowly varying waves’ satisfying

$$\frac{\partial}{\partial u} [p - p_i p^i/r_1^2 - \gamma^2(u)/r_1^2 r_5^2] \ll [p - p_i p^i/r_1^2 - \gamma^2/r_1^2 r_5^2]^{3/2}, \quad (2.14)$$

a good approximation is

$$\sigma = \frac{1}{r_1 r_5} \sqrt{p(u) - p_i(u) p^i(u)/r_1^2 - \gamma^2(u)/r_1^2 r_5^2}. \quad (2.15)$$

In the case of a small (and slowly varying) deviation from the stationary solution $p(u) = p$, $p_i(u) = 0$, $\gamma = 0$, the average value $\bar{\sigma}$ of σ reduces to

$$\begin{aligned} \bar{\sigma} = \frac{1}{r_1 r_5} \sqrt{p} \left(1 + L^{-1} p^{-2} \int du \left[p \delta p(u) - \frac{1}{4} (\delta p(u))^2 - \frac{1}{2} p_i(u) p^i(u)/r_1^2 - \frac{1}{2} \gamma^2(u)/r_1^2 r_5^2 \right] \right. \\ \left. + O(\delta p^3) + O(p_i^4) + O(\gamma^4) \right), \end{aligned} \quad (2.16)$$

where $\delta p(u) = p(u) - p$. This is the expression which will be used in section III. It is natural to choose p such that $\int du \delta p = 0$, so that the lowest order deviation is the quadratic term. Our study of the area operator reduces to computing the fluctuations $\delta p(u)$, $p_i(u)$, $\gamma(u)$ of the charge distributions for our black string.

At this point, the question should be asked if the space of solutions given by 2.2, 2.3, 2.4, 2.6 is in fact large enough, or must be further enlarged before a meaningful comparison with the D-brane Hilbert space can be made. A conclusive statement is beyond the scope of this work, but let us at least address the most obvious possibility of considering higher multipole terms in K , A_i , and A_a . We recall first that, in order for (2.2), (2.3), and (2.4) to be a solution, the fields K , A_a , and A_i are typically constrained to satisfy some elliptic set of equations. For example, when A_i is divergence free and $A_a = 0$, we must have [21]

$$\begin{aligned} (\partial_x^2 + H_5 \partial_y^2) K &= 0 \\ (\partial_x^2 + H_5 \partial_y^2) A_i &= 0 \end{aligned} \quad (2.17)$$

where ∂_x^2 and ∂_y^2 denote the usual flat-space Laplacians in x and y . For the purposes of this discussion, let us impose the boundary conditions that K and A_i are bounded near infinity so that we do not modify the asymptotic structure of the spacetime. While the general form of these equations has not yet been derived, one expects that the rough picture developed in [21,22] will continue to hold with the fields K , A_i , and A_a being entirely determined by the various multipole moments of the charge distributions and having the property that, whenever any of the dipole or higher moments is nonzero, the solution is singular on the horizon. This was shown in [14,22] to be precisely true of the field K and a short calculation based on the result of [22] shows that, at least for the case studied there, this singularity is ‘strong’ in the sense that both the once and twice integrated curvatures also diverge at the horizon. As a result, the singularity produces an infinite distortion of any object attempting to cross the horizon².

²Recall that this was not the case for the singularity discovered in [16] associated with monopole waves.

It seems reasonable to exclude strongly singular solutions, although a fully convincing argument would have to follow from some careful analysis. In an earlier version of this paper it was stated that such an analysis would have to center on the details of the BPS D-brane states. However, new evidence [23] suggests that a study of classical solutions may be sufficient. Due to the ‘deep throat’ of the extremal black string, an inhomogeneity in the distribution of one-brane charge causes a much smaller effect on the asymptotic fields than one would naively expect when the charge is placed close to the horizon. The result is that, despite the fact that the T^4 ’s near the horizon have finite size, an inhomogeneity of finite ‘intrinsic’ strength located at the horizon produces *zero* effect on the external spacetime. We suspect that the same is true for other types of charge. Details and further results will appear in [23].

It is important to note, however, that if such higher multipole solutions *are* relevant to the D-brane ensemble, then due to their strong singularities they should dominate any discussion of the effects on an object passing through the horizon. Moreover, because the du^2 term diverges, they appear to have infinite horizon area. Thus, if such solutions are included, the expected area and curvatures in the ensemble should be vastly different than the area and curvatures of the original classical solution. We adopt the working hypothesis that all BPS D-brane states in which the fivebranes do not oscillate correspond to ‘monopole’ solutions and exclude other solutions from our study.

As a final comment, recall that we have already excluded the solutions with ‘external’ waves as they involve oscillations of the five-brane in the D-brane description. Such waves are slightly different than the ones considered here as they are associated with a charge (momentum in the x^a direction) which is a vector from the point of view of the noncompact spacetime. In contrast, the waves considered here correspond to scalar charges in this sense. Note that, if we take as our basic principle that the allowed fields are those that correspond to a constant charge density over the $S^3 \times T^4$ in the flat space limit, the corresponding external waves are *not* $SO(4)$ invariant. Thus, $l = 1$ external waves are the analogue of the monopole waves considered here, while it is only the $l \geq 2$ modes that correspond to ‘higher multipole waves.’ Since $l = 1$ external waves are not strongly singular, this is consistent with our overall point of view.

III. AREA DEVIATIONS AND FLUCTUATIONS

We now discuss the quantum D-brane states and write down an operator A on the D-brane Hilbert space which corresponds to the black hole horizon area. The curvatures will be studied in section IV. We now restrict to the regime $\sqrt{p} \ll r_1, r_5$ so that we may use the effective string model of [24,25]. Thus, the ensemble for the homogeneous black string contains those states on a single string of length $\tilde{L} = Q_1 Q_5 L$ and tension $T = 1/2\pi g Q_5$ having total momentum P . Four right moving bosonic fields and four right moving fermionic fields live on this string. However, as our goal is only to estimate the order of magnitude of certain effects and to show that they are suppressed by positive powers of Q_1 and Q_5 , it will not be necessary to keep track of all the fields. Let us consider only a single right moving bosonic field $\chi(u)$; the behavior of the fermion fields is much the same. We may think of the field χ as representing the displacement of the effective string in one of the four internal directions, say y^1 .

As a result, our system is described by the action

$$S = -\frac{1}{4\pi g Q_5} \int (\partial\chi)^2 \quad (3.1)$$

and has a momentum density

$$T_{++} = (\partial_+\chi)^2 / (2\pi g Q_5) \quad (3.2)$$

along the string and a momentum density $\partial_+\chi / (2\pi g Q_5)$ in the direction of the transverse oscillations (the y^1 direction). Here we use coordinates σ_-, σ_+ along the string worldsheet. As usual, we take the momentum density to be normal ordered³. We will use the mode expansion

$$\partial_+\chi = \frac{\pi\sqrt{2gQ_5}}{\tilde{L}} \sum_{n=-\infty}^{\infty} \alpha_n e^{-2\pi i n \sigma_+ / \tilde{L}} \quad (3.3)$$

with $[\alpha_m, \alpha_n] = m\delta_{-m,n}$.

³Due to supersymmetry, this would be unnecessary if the fermions were explicitly included.

Recalling that the effective string wraps $Q_1 Q_5$ times around the compact direction, values of the worldsheet coordinate σ_+ which differ by integer multiples of L correspond to the same value of the spacetime coordinate u . As a result, parameters in the black string solution may then be identified with quantum fields on the effective string though

$$p(u) = \frac{\kappa^2 \pi}{\tilde{L}^2} \sum_{k=1}^{Q_1 Q_5} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{-2\pi i(n+m)(u+kL)/\tilde{L}} : \alpha_m \alpha_n : \quad (3.4)$$

and

$$p_i = \frac{\kappa^2}{\sqrt{2gQ_5}} \sum_{k=1}^{Q_1 Q_5} \sum_{n=-\infty}^{\infty} \alpha_n e^{-2\pi i n(u+kL)/\tilde{L}} \quad (3.5)$$

where $::$ denotes the normal ordering.

Recall that our plan is to substitute this expression into equation (2.15), and thereby to define a quantum area operator. To do so, we will have to consider products of the $p(u)$ with itself at the same point. We choose to deal with such products by normal ordering them wherever they occur. A priori, it is not clear that this is the correct approach. However, as our goal is to compute statistical effects (and not quantum effects), we feel that such a treatment is normal ordering is sufficient for this purpose.

To show that only small corrections result from taking the u -dependence into account, we need only study quadratic combinations of these operators in the appropriate ensemble. The ensemble to be used is the usual one placed in correspondence with the stationary black hole: the microcanonical ensemble with total momentum $P = P_z$. Rather than explicitly compute $\langle : \delta p^2 : \rangle$, $\langle : p_i p^i : \rangle$, $\langle : \gamma^2 : \rangle$ in the entire ensemble, we will make use of the equipartition theorem to simplify the analysis. A short derivation of the equipartition theorem in this context proceeds as follows.

Recall that the entropy of the microcanonical ensemble is of order $S(P) \sim \sqrt{Q_1 Q_5 N}$ where $N = PL/\hbar$. Let P' be the momentum of an arbitrary state in the ensemble. For large N , the ensemble with *fixed* momentum $P' = P$ is equivalent to an ensemble which allows any state with momentum $P' \leq P$; such an ensemble is overwhelmingly dominated by the states with maximal momentum. Similarly, for large N , such an ensemble is equivalent to one in which states are included with a probabilistic weight proportional to $e^{-\beta P'}$, where β is chosen by fixing the peak of $e^{-\beta P' + S(P')}$ to be at $P' = P$. Since the total momentum P' of the state is just the sum of the momenta carried by the individual modes, the weight separates into a product of weights $e^{-\beta P_n}$ for each mode n , where $P_n = \frac{2\pi n}{L} N_n$ and N_n is the occupation number of the n th mode. For any mode n with small energy and large occupation number, the classical limit applies and the momentum carried by that mode is $P_n = \beta^{-1}$, independent of n as predicted by the equipartition theorem. Higher modes receive corrections so that they carry momentum $P_n = \beta^{-1}(1 + O(1))$ and, above some cutoff n_{max} , the momentum is essentially zero. Thus, a sufficient approximation for our purposes is to set $P_n = \beta^{-1}(1 + O(1))$ for $n \leq n_{max}$ and $P_n = 0$ for $n > n_{max}$. Setting $\tilde{N} = Q_1 Q_5 N$ and taking $n N_n = n_{max} N_{n_{max}}$ for $n \leq n_{max}$, we must have $n_{max} P_n \sim n_{max}^2 N_{n_{max}} \sim \tilde{N}$. Since the cutoff must occur at some $N_{n_{max}} = \gamma^2$ of order 1, we have $n_{max} \sim \sqrt{\tilde{N}}/\gamma \sim \sqrt{\tilde{N}}$. It is perhaps reassuring to note that this result can then be used to calculate the entropy of the ensemble through $dS = \beta dP = \frac{1}{\gamma \sqrt{\tilde{N}}} d\tilde{N}$ yielding $S = \gamma^{-1} \sqrt{\tilde{N}}$. If we wish, we may then use this to maximize the validity of our approximation by choosing $\gamma = 6/(2\pi)^2$.

This result will allow us to easily estimate the difference between the actual expectation value of the area in the ensemble and the area of the stationary black string. Let us first consider the term coming from deviations in the longitudinal momentum. Expanding the momentum as

$$p(u) = \frac{1}{L} \sum_{n=-\infty}^{\infty} p_n e^{2\pi i n u / L} \quad (3.6)$$

and comparing against the stationary black string with $p = L^{-1} \int du p(u)$, we need only show that the quantity

$$\sum_{k \neq 0} \frac{\langle : p_k p_{-k} : \rangle}{\langle : p_0^2 : \rangle} \quad (3.7)$$

is small, where $p_0 = \frac{2\pi \kappa^2 \tilde{N}}{L}$. In a state with occupation numbers $\{N_n\}$, the expectation value of $p_k p_{-k}$ is

$$\langle \{N_n\} | : p_k p_{-k} : | \{N_n\} \rangle = \frac{8\kappa^4 \pi^2}{\tilde{L}^2} \left(\sum_{Q_1 Q_5 k/2 \geq n \geq 0} (n N_n) (Q_1 Q_5 k - n) (N_{Q_1 Q_5 k - n}) g_k(n) \right)$$

$$+ \sum_{n < 0} (-n) N_{-n} (Q_1 Q_5 k - n) N_{k-n} \Big) \quad (3.8)$$

where $g_k(n) = 1/8$ for $n = Q_1 Q_5 k/2$ and $g_k(n) = 1$ otherwise. Using our equipartition results that $n N_n \sim \sqrt{\tilde{N}}$ for $n \leq n_{max} \sim \sqrt{\tilde{N}}$, this gives

$$\frac{\langle : p_k p_{-k} : \rangle}{\langle : p_0^2 : \rangle} \sim \tilde{N}^{-1/2} \quad (3.9)$$

for $Q_1 Q_5 k \sim n_{max}$ while $\langle : p_k p_{-k} : \rangle \sim 0$ for $k \gg n_{max}/Q_1 Q_5$. As a result, fluctuations in the longitudinal momentum cause a deviation from the area of the stationary black hole by a fractional amount:

$$\sum_k \frac{\langle : p_k p_{-k} : \rangle}{\langle : p_0^2 : \rangle} \sim \frac{1}{Q_1 Q_5}. \quad (3.10)$$

The contributions from the $: p_i p^i :$ and $: \gamma^2 :$ terms are even smaller. Although the angular momentum is carried entirely by the fermions, the calculation of $\langle \gamma^2 \rangle$ remains much the same. Instead of the equipartition theorem, the Fermi sea approximation of $N_n = 1$ for $n \leq n_{max}$, $N_n = 0$ for $n > n_{max}$ is useful in this context. The results are

$$\frac{\langle : p_i p^i : \rangle}{p r_1^2} \sim \frac{Q_1}{(Q_1 Q_5)^2}, \quad \frac{\langle : \gamma^2 : \rangle}{p r_1^2 r_5^2} \sim \frac{1}{Q_1 Q_5}. \quad (3.11)$$

This sort of argument can also be used to justify our use of the slowly varying approximation (2.14) for the area. Another calculation along the lines of those above yields

$$\langle : \dot{p}^2 : \rangle \sim \frac{(\kappa^2 \pi)^2 N^3}{L^6 Q_1^2 Q_5^2} \quad (3.12)$$

while

$$\frac{\langle : \dot{p}_i \dot{p}^i : \rangle}{r_1^2} \sim \frac{\kappa^2 N^2}{L^4 (Q_1 Q_5)^2 Q_5}, \quad \frac{\langle : \dot{\gamma}^2 : \rangle}{r_1^2 r_5^2} \sim \frac{\kappa^2 N^2}{L^4 (Q_1 Q_5)^2} \quad (3.13)$$

so that $\dot{p}_{rms} \gg \frac{\langle : \gamma \dot{\gamma} : \rangle}{r_1^2 r_5^2} \gg \frac{\langle : p_i \dot{p}^i : \rangle}{r_1^2}$ and

$$\frac{\dot{\sigma}_{rms}}{\sigma^2} \sim \frac{1}{\sqrt{Q_1 Q_5}} \ll 1. \quad (3.14)$$

We shall not explicitly compute the fluctuations in the area operator, but we note that our computations of $\langle \delta p^2 \rangle$, $\langle p_i p^i \rangle$, and $\langle \gamma^2 \rangle$ are already measures of the size of fluctuations in our ensemble, and we have $\delta \sigma_{rms} \sim \langle \sigma \rangle / \sqrt{Q_1 Q_5}$. In this sense then, the charge distributions have only small fluctuations. As a result, a typical state has momentum density $p + O(\delta p_{rms})$ and an area extremely close to that of the stationary black string.

IV. THE SIZE OF THE SINGULARITIES

The discussion so far has focused on the ‘area operator’ that one might define by comparison with the classical solutions (2.2). This quantity was of interest both because of its prominence in the discussion of black hole entropy and because it was sensitive to short wavelength fluctuations. There is, however, another characteristic of the black string solutions which displays an even more startling sensitivity to inhomogeneities – this is the curvature at the black string horizon. As shown in [16], the curvature components in a parallel propagated orthonormal frame diverge at the horizon whenever the BPS black string is not exactly translationally invariant. Thus, if one were to define a ‘horizon curvature operator’ in the D-brane Hilbert space by looking at the dependence of the horizon curvature on the asymptotic momentum distributions $p(u)$, $p_i(u)$, $\gamma(u)$, it would diverge on all states in the microcanonical ensemble, in marked contrast to the finite curvature of the stationary black hole solution. Does it follow that any object approaching a ‘real’ macroscopic black string would be ripped apart by the large curvatures present in its microscopic quantum description?

In fact, it was already shown in [16] that the singularity at the horizon is of a relatively mild sort. Although the curvature itself diverges, the total distortion of, say, a set of uncharged test particles passing through the horizon is given by the twice integrated curvature which is finite. We shall also see below that the relative velocities induced between two such test particles remain bounded as the horizon is approached. Thus, it is natural to ask the following questions: 1) For freely falling test particles, how do the relative velocities that two dust particles obtain from the inhomogeneous black string compare with those imparted by the stationary black string? How does the total tidal distortion caused by the inhomogeneities compare with the distortion that would be induced by a stationary black string? 2) We may also wish to consider test particles which are not freely falling, but instead remain at a constant value of the radial coordinate outside the horizon⁴. This is the worldline followed by an extremal test particle with the same sign of the charge as the black hole. We should therefore ask, “how close does such a test particle have to be to the horizon before the divergent tidal force from the wave becomes comparable to the tidal forces present in the stationary black hole?” Again using the asymptotic charges to map the appropriate functions on the space of classical solutions to operators on our D-brane Hilbert space, we will find that, for the microcanonical ensemble which corresponds to the stationary black hole, the effects from the waves are smaller than the stationary effects by powers of $1/Q_1Q_5$. Since we use the results of [16], for the rest of this section we shall restrict ourselves to the case $r_1 = r_5 = r_0$, $\gamma(u) = 0$, $p_i(u) = 0$, considered there. While we must restrict to the case $\sqrt{p} \ll r_1, r_5$ to use the fluctuation results from section III, this approximation has not been explicitly used below; all classical terms of any order in \sqrt{p}/r_0 have been kept.

A. Comparing Curvatures

Let us begin by answering the second question – just how close to the horizon must we be for the curvature caused by waves to become comparable to the curvature of the stationary black hole? When written in terms of the one-form basis used in [16], the divergent curvature terms have the form

$$R \sim \frac{\dot{\sigma} r_0^2}{r^2} \sim \frac{\dot{p}}{r^2 \sqrt{p}} \quad (4.1)$$

where we have restored the dimensional factors and translated the expression in [16] into the coordinates used here. We must, of course, know something about the curvatures produced by a stationary black hole as well. It turns out [26] that, for the stationary black hole, the leading order term near the horizon is p/r_0^4 . Again using the root mean square value of \dot{p} , the term from the inhomogeneities dominates the stationary curvature only when

$$r \sim \left(\frac{\dot{p}_{rms}}{p^{3/2}} r_0^4 \right)^{1/2} \sim r_0 (Q_1 Q_5)^{-1/4}. \quad (4.2)$$

Let us denote this value of r by r_A . While we have derived this result using the curvature components in a particular basis, we may expect a similar result in general as both the wave-induced and stationary parts of the curvature are associated with curvature components proportional to the same null one-form du .

It is interesting to note that, when written in terms of the charges and moduli, $r_0 = (Q_1 Q_5)^{1/4} \sqrt{g}/V^{1/4}$. Thus, the transition occurs at

$$r \sim r_A = \sqrt{g} V^{-1/4} \quad (4.3)$$

where factors of the string length have been set to one. For the classical supergravity description to be valid, we should have small coupling and large T^4 so that $r_A \ll l_{string}$. Although $r = r_A$ is still an infinite proper distance from the horizon, the placement of a particle at $r = r_A$ requires extreme care. For example, the corresponding redshift from infinity is r_A/r_0 , so an uncharged particle dropped in from infinity would have to shed $(1 - r_A/r_0)$ of its mass to instantaneously come to rest at $r = r_A$. Also, while it is not clear to what extent quantum effects become relevant outside the horizon of an extremal black hole⁵, if they are relevant anywhere, they are likely to be relevant at such a value of r .

⁴Thanks to Rob Myers for raising this issue.

⁵In particular, it is interesting to note that, for an extremal black *hole*, the proper acceleration required for a test particle to travel along an orbit of the Killing field $\frac{\partial}{\partial t}$ remains bounded and approaches an asymptotic value of order $1/r_0$ as $r \rightarrow 0$. This

B. Curvature Effects

Let us now consider a set of test particles whose worldlines are not so finely tuned, but which fall across the horizon on geodesics. We would like to compute the relative velocity and total distortion induced by the inhomogeneities. Since the stationary curvature terms dominate the terms from the time dependence outside $r = r_A$, any distortion produced by the inhomogeneities before reaching this value of r will be negligible compared to that induced by the stationary terms. As a result, we need only follow our test particles from r_A to the horizon.

The relative velocity v induced between two test particles separated by a small distance l is given by integrating the curvature once along the worldline. This is most easily accomplished by using coordinates (U, V, q, θ) based on those of [14,16]; details of the coordinate system are given in appendix A. What is important about this coordinate system is that the horizon is just the surface $U = 0$ and that the metric is C^0 there. The coordinates θ on the three-sphere are just the same as those used above. Furthermore, U is a function only of the old coordinate u :

$$U = -\frac{1}{2} \int_u^{+\infty} \frac{du}{G^2} \text{ where} \\ G = e^{\frac{1}{2} \int_0^u \sigma(u) du}. \quad (4.4)$$

Let us parameterize the worldline by the coordinate U . Because the connection coefficients are bounded, the ratio $dU/d\lambda$ is C^0 at $U = 0$ for any affine parameter λ along the worldline. Suppose that our geodesic intersects the horizon at the point (U, V, q, θ) . Since the curvature is continuous in q, V, θ^i , when our geodesic is close to the horizon we may approximate the curvature by setting these coordinates equal to their value where the worldline crosses the horizon. Thus, $v/l \sim \int_U^0 R dU$.

Whether this approximation remains valid at the finite distance U_A from the horizon is a more subtle question due to the multiplicity of length scales r_0, \sqrt{p}, L which appear in the classical metric. Recall, however, that due to the periodic identifications, there are many values of q corresponding to any given point on the horizon; going once around the z direction rescales q by $e^{-2\pi L}$. If we use a coordinate patch in which q takes values less than p/r_0 , then appendix A shows that the part of a generic geodesic between $r = r_A$ and the horizon can be approximated by taking U to be an affine parameter and U, V, q, θ^i to be constant. The qualification ‘generic’ arises because this is in fact not true of those geodesics which are very close to the integral curves of the killing field $\frac{\partial}{\partial V}$. Such integral curves never cross the horizon (they ‘run directly away from the horizon at the speed of light’), and nearby geodesics can circle the compact direction many times before finally crossing the horizon (so that, for example, q is not constant along those trajectories). Such geodesics must, however, be very carefully chosen: given any C^0 timelike vector field there is a notion of what fraction of the geodesics fired from a given point fail to have the desired property and this fraction vanishes for large $Q_1 Q_5$. Considering such finely tuned geodesics can thus be thought of as a higher order correction. The class of generic geodesics includes *all* geodesics which fall toward the black string from an initial position r of order r_A or larger.

As a result, the typical relative velocity imparted to test particles with a small separation l during their fall from $U = U_A$ to some $U = U_B$ near the horizon is given by

$$v_{wave}/l \sim \int_{U_A}^{U_B} \frac{\dot{\sigma} r_0^2}{r^2} dU \\ = -\frac{4q}{r_0} \int_{u_A}^{u_B} \dot{\sigma} du \\ = -\frac{4q}{r_0} [\sigma(u_B) - \sigma(u_A)] \quad (4.5)$$

where u_A, u_B are the corresponding values of the coordinate u . While this does not have a well defined limit as $u_B \rightarrow \infty$, it is clearly bounded by

$$|\frac{v_{wave}}{l}| < (const) \frac{q}{r_0^3} \frac{|\delta p|}{\sqrt{p}} \quad (4.6)$$

observation was also made in [27]. Similarly, a geodesic which is initially tangent to $\frac{\partial}{\partial t}$ requires a proper time of order r_0 to fall across the horizon, even if it begins its journey at very small r . Due to its ergosphere, a black string with momentum is more complicated, but it remains true that a test particle requires only an acceleration of order $1/r_0$ to remain at any value of r .

where $|\delta p|$ is the scale of variation of $p(u)$. Taking $\frac{|\delta p|^2}{p^2} \sim \frac{|\delta p_{rms}|^2}{p^2}$, we may compare this bound with the relative velocity produced by the stationary term:

$$v_{stat}/l \sim \sigma^2 |U_A| = \frac{2qr_A^2 \sqrt{p}}{r_0^5}. \quad (4.7)$$

Thus, the ratio of these effects is just

$$\frac{v_{wave}}{v_{stat}} \sim \frac{\delta p_{rms} r_0^2}{r_A^2 p} \sim 1 \quad (4.8)$$

and, even in the small region where the wave-induced curvature is larger than the stationary term, it produces only about the same total effect on the distribution of test particles. If the particles begin their fall from rest relative to each other at $r = r_A$, the ratio of the distortions is also comparable. This relative velocity and distortion are, however, only small fractions of the total effect produced by the stationary term if the test particles are dropped from $r \gg r_A$. In this sense then, the effect of the singular term is for a typical D-brane state.

V. DISCUSSION

We have seen that the ensemble is characterized by small fluctuations (order of $\frac{1}{(Q_1 Q_5)^{1/2}}$ or smaller) in the charge distributions $p(u)$, $p_i(u)$, and $\gamma(u)$ and that the charge distributions of a typical string are very close to that of the stationary black string. As a result, the D-brane ensembles provide only a small statistical correction to the horizon area and curvature of a stationary black string, and a typical quantum state corresponds well to the classical solution. A similar analysis could be performed for ensembles [6] corresponding to rotating black holes or [14,15] for a black string with nontrivial distributions of charge. The analysis of the horizon area should be essentially the same in such cases, through the curvature analysis would differ for the nonuniform case as the classical black string has a singular horizon whenever the charge distributions are nonuniform.

A few comments about the nature of these singular horizons are now in order. As noted in [16], the singularities are ‘weak’ in the sense that they are twice integrable and so produce finite distortion of any object falling through the horizon. In addition, we note that the connection coefficients are bounded (see appendix) so that geodesics crossing the horizon have well defined tangent vectors at $U = 0$ and can be uniquely continued beyond this surface. Note that this is true even if the wave σ for $U < 0$ (outside) is quite different from the wave σ for $U > 0$ (inside). Such singularities are also *dynamically* benign as these spacetimes represent exact solutions of classical string theory to all orders in α' [28] and the dilaton remains small. Thus, considering such singular solutions may be physically reasonable.

The reader may note that we have not yet discussed deviations of the cross sections for absorption of low energy quanta. This is because no new calculations are needed for this purpose. The cross section for absorption of a low energy quantum of energy ω has been shown [7–9] to be given (in the appropriate regime) by

$$\frac{A_H}{T_H} \omega (\langle N_\omega \rangle + 1) \quad (5.1)$$

where A_H and T_H are the horizon area and Hawking temperature of the original stationary black hole, and $\langle N_\omega \rangle$ is expectation value of the occupation number of the mode with frequency ω in the particular D-brane state. For a low energy quantum, a typical state has an occupation number $T_H/\omega(1 + O(1))$ as determined by the equipartition theorem. While the fluctuations of N_ω for a single mode are of the same order as the expectation value, if we consider an incoming particle with a frequency spread much greater than the gap associated with the black hole ($\Delta\omega \gg \frac{\hbar c}{L} = \omega_{gap}$) it is appropriate to average over many modes. A typical D-brane state will then be associated with a cross section

$$A_H (1 + O(\sqrt{\frac{\Delta\omega}{\omega_{gap}}})) \quad (5.2)$$

for $\omega/T_H \ll 1$. Here, the $O(\sqrt{\frac{\Delta\omega}{\omega_{gap}}})$ term represents the fluctuations.

One difficult remaining question involves the status of the singular solutions associated with higher multipole moments ($l \geq 1$ for the waves considered here, $l \geq 2$ for external waves). Classically, one may argue against such

solutions due to their strong singularities. However, a conclusive treatment has yet to be given. One may hope to argue that the higher multipole moments of such states vanish exactly. Some evidence for this will be presented in [23].

Finally, some comments are in order concerning the possibility of extending this work to non-BPS states. Such a project would seem to be quite difficult, not only due to the decreased protection from supersymmetry, but also because of the fact that non-BPS classical solutions in general lack the null translational symmetry which allowed us to assign conserved charge *distributions* to each solution and to in fact characterize each black string by its associated distribution of charge. Thus, the consideration of non-BPS states destroys the bridge used here to connect the classical and quantum descriptions.

ACKNOWLEDGMENTS

It is a pleasure to thank Gary Horowitz, Juan Maldacena, Samir Mathur, Cristina Marchetti, and Rob Myers for many helpful discussions. Special thanks are due to Haisong Yang for supplying the unpublished stationary curvature near the horizon. I would also like to thank Sumati Surya for her work in deriving related results, and Simon Ross for pointing out an error in the Appendix in an earlier version of this work.

APPENDIX A: GEODESICS NEAR THE HORIZON

In section IV, it was stated that along ‘generic’ geodesics passing between U_A and $U = 0$ the coordinates q, V, θ are approximately constant and U is effectively an affine parameter. The purpose of this appendix is to derive this result. Let us begin by recalling the details of the metric and coordinate system. Here, we continue to specialize to the case $r_1 = r_5 = r_0$, $p_i(u) = 0$, $\gamma(u) = 0$, but we restore the dimensional factors that were suppressed in [16]. Below, we have adjusted the normalizations so that all three coordinates (U, V, q) have dimensions of length. Note that our coordinate V coincides with the V coordinate of [16], which was called ν in [14]. Also note that our coordinate r is the r coordinate of [14] (which vanishes on the horizon) and not the r coordinate of [16].

The coordinates U, V, q are defined by

$$\begin{aligned} G(u) &= e^{\int_0^u \sigma du} \\ U &= - \int_u^{+\infty} G^{-2} du \\ W &= \frac{Gr}{r_0 \sqrt{r^2 + r_0^2}} \\ q &= -\frac{1}{2r_0 W^2} - 3r_0 \int_0^U \sigma dU \\ V &= v - \frac{\sigma r_0^4}{r^2} - 2r_0^2 \int_0^u \sigma^2 du + 3r_0^4 \int_0^U \sigma^2 W^2 dU \end{aligned} \quad (A1)$$

where the integrals dU are performed along contours of constant q, V, θ . The metric is then

$$\begin{aligned} ds^2 &= -r_0^2 W^2 dU dV + r_0^4 \sigma^2 W^4 r^2 (1 + 8r^2/r_0^2 + 4r^4/r_0^4) dU^2 \\ &\quad + \left[2 \frac{\sigma}{r_0} W^4 r^2 (r^2 + r_0^2) (2r^2 + 3r_0^2) + 6W^2 r_0^7 \int_0^U \sigma^2 W^4 dU \right] dU dq \\ &\quad + r_0^{-2} W^4 (r^2 + r_0^2)^3 dq^2 + (r^2 + r_0^2) d\Omega_3^2. \end{aligned} \quad (A2)$$

Now, for the case of interest, we have shown that the momentum density is ‘slowly varying’ so that the approximation $\sigma = \sqrt{p(u)}/r_0^2$ is valid. Furthermore, we have seen that the variations δp in the momentum density $p(u)$ are small compared to the average value p . Under these circumstances, the form of the metric greatly simplifies near the horizon. Let us suppose that $\epsilon = r/r_0 \ll 1$. Then the corresponding value of U is $U = -\frac{\epsilon^2}{W^2 \sqrt{p}} (1 + O(\epsilon^2))$. As a result, to leading order we have $W^2 = -\frac{1}{2qr_0}$ so that W is constant along the contours of integration in (A1) and $U_A = \frac{qr_0 \epsilon^2}{\sqrt{p}}$. Expanding the various terms, we may write the metric as

$$ds^2 = -r_0^2 W^2 dU dV + r_0^6 W^4 \sigma^2 \epsilon^2 dU^2 + \epsilon^2 \sigma W^4 r_0^4 dq dU + W^4 r_0^4 dq^2 + r_0^2 d\Omega_3^2 + O(\epsilon^4) + O\left(\frac{\delta p}{p}\right), \quad (A3)$$

which is a much more convenient expression to work with. It is important to keep track of the ϵ^2 terms as, if we take a derivative with respect to U , $\frac{\partial \epsilon^2}{\partial U}$ is of order 1.

If we now consider a null or timelike geodesic which runs from U_A to the horizon, we can use causality to bound the change in q along this path. If we parameterize the geodesic by the coordinate U , the tangent vector $n^\alpha = dx^\alpha/dU$ must satisfy

$$n^U n^V > W^2 r_0^2 (n_q)^2 \quad (\text{A4})$$

plus corrections of order ϵ^2 . Let us introduce the parameter $n^U/n^V = \alpha^2$. Then we may bound n^q in terms of n^U . This means that the change δq in the coordinate q along the geodesic must satisfy

$$\delta q < \frac{U_A}{\alpha W r_0} \sim \frac{q^{3/2} r_0^{1/2} \epsilon^2}{\alpha \sqrt{p}}. \quad (\text{A5})$$

Choosing $q < p/r_0$ yields $\delta q \sim q \epsilon^2 / \alpha$, so that unless $\alpha \sim \epsilon^2$ the coordinate q changes by only a very small amount along the geodesic.

Let us allow $\delta q \sim q$, or $\alpha \sim \epsilon^2$. We would like to say that the excluded geodesics are but a small fraction of the whole. An obvious difficulty is the lack of a Lorenz covariant normalized measure on the space of timelike and null vectors. Let us, however, suppose that we fix an arbitrary normalized measure on $[1, 0)$. Then, given a normalized future directed timelike vector t^α at x , we may use this vector to define a measure on the space of future directed timelike geodesics through x , say, by pull-back through $-t \cdot v$ where v is the unit tangent to the geodesic. Then, one statement that can be made is that any normalized timelike C^0 vector field on the stationary black string spacetime must differ from $|\frac{\partial}{\partial U} + \frac{\partial}{\partial V}|^{-1}(\frac{\partial}{\partial U} + \frac{\partial}{\partial V})$ by some fixed finite boost on the horizon. As a result, if we take the limit $Q_1, Q_5 \rightarrow \infty$ while holding the vector field fixed, the measure assigned to the set of excluded geodesics vanishes in that limit, no matter what vector field was chosen. In this sense, then, consideration of the excluded geodesics is equivalent to a higher order correction. It is comforting to note that, in terms of the original coordinates (u, v, r) and any parameter λ , we have

$$\frac{dU}{d\lambda} = G^{-2} \frac{du}{d\lambda} \quad (\text{A6})$$

$$\frac{dq}{d\lambda} = -\frac{q}{r_0} \frac{d}{d\lambda} \ln(r/r_0) - q \sigma \frac{du}{d\lambda} + O(\epsilon^2) \quad (\text{A7})$$

so that any geodesic which begins at $r \geq r_A$ with $dr/d\lambda \leq 0$ has $\frac{dq}{dU} \sim \frac{1}{\epsilon^2 W r_0}$ or smaller (corresponding to $\alpha \sim \epsilon^2$ or larger) and has been included in our analysis.

The last statement to be verified is that U may be treated as an affine parameter along the geodesic all the way from the horizon to $U = U_A$. To do so, let us first note that (since $q < p/r_0$) all of the connection coefficients $\Gamma_{\beta\gamma}^U$ are of order ϵ^2 except for $\Gamma_{UU}^U \sim \sqrt{p}/r_0 q$ and $\Gamma_{qU}^U \sim r_0 W^2$. As a result, since $\dot{q} \sim \epsilon^2$, the geodesic equation says that if \dot{U} denotes the derivative of U along the worldline with respect to some affine parameter, then $d\dot{U}/\dot{U} \sim \sqrt{p}/r_0 q dU$. Thus the total fractional change in \dot{U} is on the order of $\sqrt{p}/r_0 q U_A = 2\epsilon^2$ and we may treat U as an affine parameter for $0 > U > U_A$.

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